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Finite Element Analysis of Two-Dimensional Slow Non-Newtonian Flows

A finite element method is applied to isothermal incompressible two-dimensional slow flows of power-law fluids. Examples considered are rectangular and axisymmetric converging channel flows, recirculating flows in rectangular channels, and flow round cylinders. Results are successfully compared with both finite difference and analytical solutions. The flexibility of finite element methods makes them very suitable for problems involving complex boundary geometries. The method used is particularly suitable for non-Newtonian flows and can treat both rectangular and axisymmetric geometries.

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SCOPE

There are many industrial examples of slow non-Newtonian flows, such as in polymer processing which involves the flow of highly viscous melts in extruders, dies, and moulds. In slow flows, fluid inertia forces are negligible compared to viscous and pressure forces.

Attention is confined to steady incompressible two-dimensional viscous flows in regions whose boundaries are rigid but not necessarily stationary. The flows are two-dimensional in the sense that only velocities in the plane of the problem region are considered, although this does not preclude flow normal to this plane. The flow may be axisymmetric in that the actual three-dimensional flow region is formed by rotating the two-dimensional problem region about some axis of symmetry in the same plane. The examples considered here are flows in both rectangular and axisymmetric converging channels which are typical of polymer extrusion dies, including wire-coating dies. Also considered are recirculating flows in rectangular channels, such as screw extruder channels, and flow round cylinders, such as mixing pins in these channels. The flows are assumed to be isothermal in the sense that temperature variations do not significantly affect the viscous properties of the fluids.

The object of the work described here is to apply finite element (FE) methods to two-dimensional flow problems traditionally solved by finite difference (FD) methods and to demonstrate the advantages of the FE approach. FD methods involve substituting FD approximations for the derivatives contained in the differential conservation equations. These approximations are in terms of the values of the variables at discrete points in the problem region. The points usually lie in rows parallel to the coordinate axes and are frequently, though not necessarily (Gosman et al., 1969) equispaced along these rows. Examples of FD analysis of two-dimensional flow problems include Burggraf (1966), Gosman et al. (1969), and Martin (1969).

FE methods require the problem region to be divided into small subregions, or finite elements. For example, a two-dimensional region may be divided into a mesh of triangles or quadrilaterals. The relevant variables are again required in terms of values at discrete nodal points in the mesh, the corners of the elements (and sometimes additional points on the element boundaries). An appropriate (polynomial) function of position is assumed for each variable within each element. A variational formulation (see, for example, Schechter, 1967) is normally used to solve for the nodal point variables by minimizing appropriate functionals. FE methods are well established in the fields of structural and solid continuum mechanics (see Zienkiewicz, 1971) but have not been widely used in solving fluid mechanics and heat transfer problems. General examples include Martin (1968) and Oden and Somogyi (1969). Atkinson et al. (1969, 1970) applied FE methods to two-dimensional Newtonian flow problems. In a previous paper (Palit and Fenner, 1972) the present authors applied a FE method to non-Newtonian channel flow normal to the plane of the mesh.

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CONCLUSIONS AND SIGNIFICANCE

The applicability of a FE method to the analysis of two-dimensional slow non-Newtonian flows is demonstrated. Where possible the numerical results are successfully compared with both FD and analytical solutions. The FE method used is particularly convenient for treating non-Newtonian flows as it implies the approximation of constant viscosity over each element. A further advantage is that axisymmetric flow problems can be solved with only trivial modifications to the method for rectangular flows. The boundary conditions for velocities required in the FE method are simpler to apply than those for vorticity used in many FD methods. The FE method is also much more flexible in the type of boundary geometries it can handle. Although the individual triangular elements should be reasonably equilateral, meshes of such elements can be constructed to handle any irregular boundary shape.

The results presented for converging channel flows, recirculating flows in rectangular channels, and flow round cylinders serve to illustrate typical applications of the FE method rather than to provide exhaustive studies of particular problems. The results for converging channels are compared in terms of flowrates and pressure differences with analytical solutions for fully developed flow, and provide examples of flow geometries that are not easily handled by FD methods. The results for recirculating flows are compared in terms of stream function profiles with a FD solution. The distribution of flow round cylinders is included as an example of the complex geometries that can be handled by the FE method. Thus FE methods are potentially very powerful tools for the solution of engineering problems of fluid flow.

FORMULATION OF THE PROBLEM

Figures 1 and 2 show the geometries, coordinates, and boundary velocities for two-dimensional converging and recirculating channel flows. Figure 1 with $V = 0$ is typical of the flow regions in many types of polymer extrusion dies (see, for example, Bernhardt, 1959) and with $V > 0$ is typical of wire coating dies (Fenner and Williams, 1967). In the latter case, and often in the former too, the flow is also axisymmetric. Figure 2 is typical of the recirculating flow region in extruder screw channels (see, for example, Fenner, 1970). For steady incompressible slow flows the differential stress balance equations (derived from momentum conservation) in Cartesian coordinates are, in terms of pressure and viscous stress gradients,

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{1}$$

$$\frac{\partial p}{\partial y} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \tag{2}$$

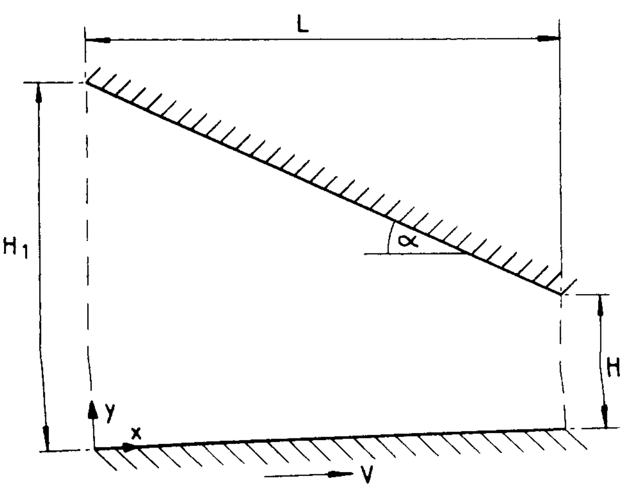

Fig. 1. Converging channel flow geometry and coordinates.

In order to automatically satisfy mass conservation in incompressible two-dimensional flows, it is convenient to introduce a stream function ψ (see, for example, Burggraf, 1966; or Gosman et al., 1969) from which the velocity components may be derived as

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \tag{3}$$

Neglecting the cross-viscosity term in the general constitutive equation for a Stokesian fluid (Fenner, 1970) the viscous stress components become

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x}, \quad \tau_{yy} = 2\mu \frac{\partial v}{\partial y}, \quad \tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{4}$$

The omission of cross-viscosity effects is unavoidable owing to the lack of adequate viscous property data (Fenner, 1970). Similarly, it must be assumed that the viscosity is a function only of the second invariant of the rate of deformation tensor:

$$\mu = \mu(I_2), \quad 4I_2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \tag{5}$$

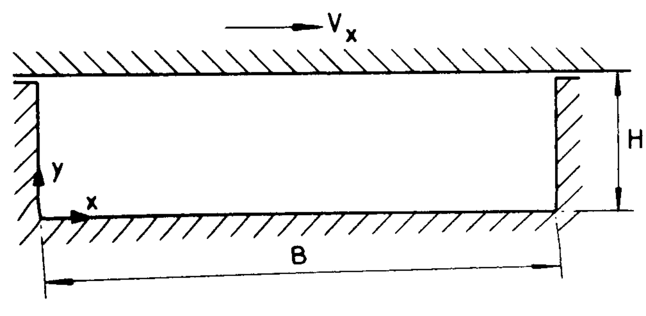

Fig. 2. Recirculating channel flow geometry and coordinates.

Although any functional form of $\mu(I_2)$ can in principle be handled attention is confined to a power-law form which is applicable to many non-Newtonian fluids, particularly polymer melts (Fenner, 1970):

$$\mu = \mu_0 \left(\frac{\sqrt{4I_2}}{\gamma_0} \right)^{n-1} \tag{6}$$

where μ_0 is the effective viscosity at the reference shear rate γ_0, and n is the power-law index.

The usual procedure in FD methods is to eliminate the pressure from Equations (1) and (2) by differentiation and to express the stresses in terms of ψ. The result is a fourth-order equation in ψ which for Newtonian flow reduces to $\nabla^4\psi = 0$. While this equation can be approximated by FD expressions it is more usual to use the equivalent pair of second-order equations formed by the introduction of a vorticity ω (see, for example, Burggraf, 1966; Gosman et al., 1969; Martin, 1969), which for Newtonian flow become

$$\nabla^2\psi = \omega, \quad \nabla^2\omega = 0 \tag{7}$$

Equations (1) to (5) are applicable only in Cartesian coordinate systems. Equivalent, though rather more complicated, equations in cylindrical polar coordinates are required for the analysis of axisymmetric flows by FD methods (Dyer, 1969).

Finite Element Formulation

Numerous FE methods are possible involving elements of various shapes and complexities (Zienkiewicz, 1971; Atkinson et al., 1969, 1970), but only one method which is particularly convenient for non-Newtonian flow problems is considered in detail here. A mesh of triangular elements with nodal points at the corners is selected as being simple to use and readily applicable to arbitrary boundary geometries. Figure 3 shows a typical element, numbered m, in the x-y plane (local coordinates parallel to the channel coordinates in Figures 1 and 2).

The main variable to be considered is the stream function, defined by Equation (3). A suitable polynomial function of position within each element is assumed for ψ. For example, in treating downstream channel flow Palit and Fenner (1972) assumed a linear function for velocity which resulted in constant velocity gradients and hence constant viscosity over each element. To achieve the same

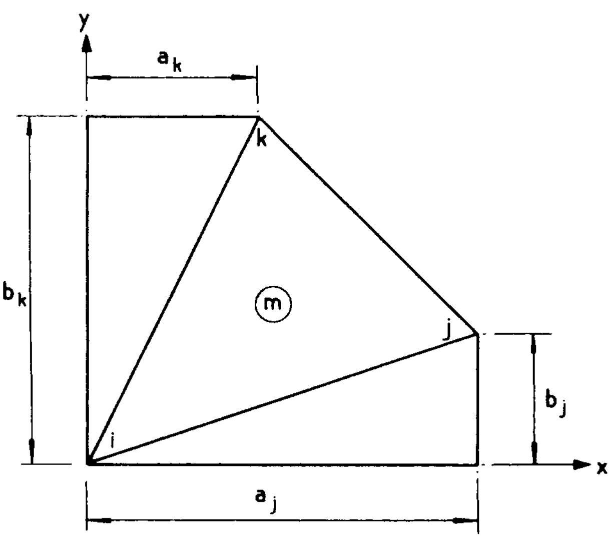

Fig. 3. Triangular element dimensions.

effect here ψ should be a quadratic function:

$$\psi = F(x, y) = C_1 + C_2x + C_3y + C_4x^2 + C_5y^2 + C_6xy \tag{8}$$

$$u = -\frac{\partial F}{\partial y} = -F_y(x, y) = -C_3 - 2C_5y - C_6x \tag{9}$$

$$v = \frac{\partial F}{\partial x} = F_x(x, y) = C_2 + 2C_4x + C_6y \tag{10}$$

$$\frac{\partial u}{\partial x} = -C_6, \quad \frac{\partial u}{\partial y} = -2C_5, \quad \frac{\partial v}{\partial y} = C_6, \quad \frac{\partial v}{\partial x} = 2C_4 \tag{11}$$

The six constants must be expressed in terms of nodal point variables ψ_i, ψ_j, ψ_k, and three other variables. Atkinson et al. (1969, 1970) used values of ψ and its first derivatives with respect to x and y, a total of nine variables, together with a modified cubic function for ψ. Such a formulation is much less convenient for non-Newtonian flow. In order to reduce the number of variables to six but to still retain both ψ derivatives, and hence both velocity components, a new variable is defined:

$$\varphi = u + v = \frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y} \tag{12}$$

Hence the constants in Equation (8) may be expressed in terms of the nodal point values of ψ and φ:

$$C_r = c_{r1}\,\psi_i + c_{r2}\,\varphi_i + c_{r3}\,\psi_j + c_{r4}\,\varphi_j + c_{r5}\,\psi_k + c_{r6}\,\varphi_k \tag{13}$$

where the c_{rs} are determined by the element geometry (that is, by a_j, b_j, a_k and b_k), and are obtained by solving the six equations:

$$\begin{aligned} \psi_i &= F(0, 0) \quad , \quad \varphi_i = F_x(0, 0) - F_y(0, 0) \\ \psi_j &= F(a_j, b_j) \quad , \quad \varphi_j = F_x(a_j, b_j) - F_y(a_j, b_j) \\ \psi_k &= F(a_k, b_k) \quad , \quad \varphi_k = F_x(a_k, b_k) - F_y(a_k, b_k) \end{aligned} \tag{14}$$

The condition to be satisfied by the nodal point variables is that the total viscous dissipation over the (three-dimensional) flow region should be a minimum (Schechter, 1967; Atkinson et al., 1969, 1970). Such a variational statement is equivalent to the equilibrium condition expressed by Equations (1) and (2). The viscous dissipation functional is

$$f(\psi, \varphi) = 2\mu I_2 \tag{15}$$

The integral to be minimized with respect to the nodal point variables is therefore

$$\chi = \iiint f \cdot d\,(\text{volume}) \tag{16}$$

which is the sum of the integrals over the individual elements.

Rectangular Geometry

For the rectangular case the contribution of the typical element is

$$\chi^{(m)} = W \iint 2\mu I_2 \cdot d\,(\text{area}) \tag{17}$$

$$= 2W\mu_m\Delta_m\,(C_6^2 + (C_4 + C_5)^2) \tag{18}$$

where W is the width of flow normal to the x-y plane and Δ_m is the area of the element. The condition to be satisfied for every nodal point variable such as ψ_i is

$$\frac{\partial\chi}{\partial\psi_i} = 0 = \Sigma_m \frac{\partial\chi^{(m)}}{\partial\psi_i} \tag{19}$$

$$= \Sigma_m \mu_m \Delta_m \left(C_6 \cdot \frac{\partial C_6}{\partial \psi_i} + (C_4 + C_5) \left(\frac{\partial C_4}{\partial \psi_i} + \frac{\partial C_5}{\partial \psi_i} \right) \right) \quad (20)$$

where the summation need only be performed for elements which involve the point i.

Axisymmetric Geometry

The axisymmetric case is only slightly more complicated. In order to satisfy mass conservation for flow which is symmetric about an axis in its plane, the velocity components are redefined in terms of stream function as

$$u = -\frac{\overline{r}}{r} \cdot \frac{\partial \psi}{\partial y}, \quad v = \frac{\overline{r}}{r} \cdot \frac{\partial \psi}{\partial x} \quad (21)$$

where r is the radius from the axis and $\overline{r}$ is some (arbitrary) mean radius. Within each element r is assumed to be constant and is equated to r_m, the radius of the centroid of element m. This ensures that I_2 and μ are constant over the element. Thus the definition of φ is modified:

$$\varphi = u + v = \frac{\overline{r}}{r_m} \left(\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \right) \quad (22)$$

with consequent simple changes in the defining Equations (14). Also since I_2 and μ are constant the contribution of the typical element to χ is

$$\chi^{(m)} = 2\pi r_m \iint 2\mu I_2 \cdot d\,(\text{area}) \quad (23)$$

and Equation (19) becomes

$$\frac{\partial \chi}{\partial \psi_i} = 0 = \Sigma_m r_m \mu_m \Delta_m \left(C_6 \cdot \frac{\partial C_6}{\partial \psi_i} + (C_4 + C_5) \left(\frac{\partial C_4}{\partial \psi_i} + \frac{\partial C_5}{\partial \psi_i} \right) \right) \quad (24)$$

Final Equations

The final set of $2N$ equations is of the form:

$$[K]\,[\psi, \varphi] = [0] \quad (25)$$

where $[\psi, \varphi]$ is the column matrix containing the N nodal point values of ψ and φ, and $[K]$ is the symmetric viscous stiffness matrix.

Before attempting to solve Equations (25) the boundary conditions for ψ and φ must be specified. These usually involve prescribed values of ψ and φ or their first derivatives normal to the boundary. If the prescribed value of say ψ at a particular point is ψ_b then the relevant element in the zero column matrix in (25) is set to $\psi_b \cdot M$ and the corresponding diagonal element of $[K]$ is set to M, where M is a number very much larger than the nondiagonal elements of $[K]$.

Zienkiewicz (1971) has discussed the application of (first) derivative boundary conditions. In the commonly occurring case of a zero derivative normal to the boundary, for example at a line of symmetry, no change of Equations (25) is required.

Solution of the Equations

The equations can only be solved for the ψ_i and φ_i with the aid of a digital computer. An iterative successive overrelaxation method is used (see, for example, Varga, 1962), and after a prescribed number of iteration cycles (after every cycle in the results presented here) the viscosity of every element is updated using the current velocity gradients to find I_2 and μ from Equations (5) and (6).

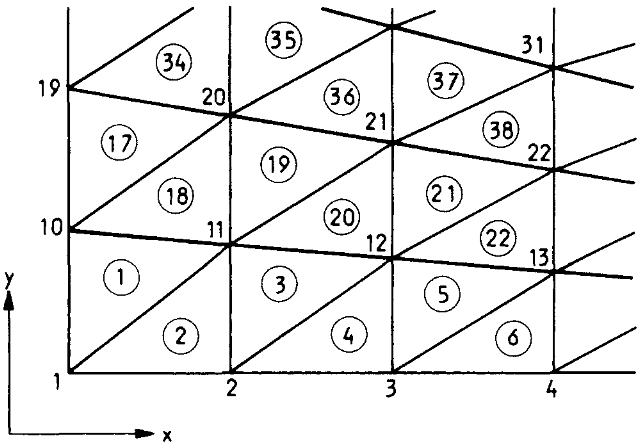

Fig. 4. Part of an 81 (9 × 9) point FE mesh for converging channel flow.

Choice of Mesh, Accuracy, and Convergence

Figure 4 shows part of the type of FE mesh used for the converging channel problem (Figure 1). Both nodal points and elements are numbered (in this case there are 81 points, 9 × 9, and 128 elements). In terms of computer programming it is convenient to start from a uniform rectangular mesh, for which the nodal point numbers and coordinates are readily generated. Once generated the mesh can then be modified to suit the particular problem. For example, for a taper angle α (Figure 1) the modified y coordinate of a nodal point originally at (x, y) can be obtained as

$$y' = y \left(1 - \frac{x \tan\alpha}{L} \right) \quad (26)$$

For non-Newtonian flow normal to the problem region the FE and FD equations are identical under the right conditions, and therefore the results are of similar accuracy (Palit and Fenner, 1972). Such a direct comparison is not possible for the present formulation for two-dimensional flow and its accuracy is best assessed by refining the mesh and by comparison with analytical solutions where possible.

Two types of convergence must be considered in the FE analysis. Firstly, as the number of elements is increased and their sizes reduced the FE solution should approach the true solution to the physical problem. Zienkiewicz (1971) has stated the criteria for this convergence. For the present problem the shape function given by Equation (8) must be capable of representing constant velocity gradients as the element size is reduced to zero and the velocities must be continuous across the element boundaries. Both of these conditions are satisfied by the linear velocity profiles and constant velocity gradients implied by Equations (9), (10), and (11).

The second type of convergence is that of the successive overrelaxation solution procedure. As the number of iterations is increased the values of the nodal point variables should tend to constant values. Palit and Fenner (1972) discussed this convergence for flow normal to the plane of the mesh and stated the sufficient condition for Newtonian flow that the elements should be acute or right angled triangles. No such simple result has been obtained for two-dimensional flows. Experience has shown, however, that the elements should be acute or right angled, and as equilateral as possible. With long thin elements the rate of convergence is much slower and the accuracy of the results is reduced. This contrasts with the downstream flow problem where overall accuracy was considerably enhanced by modifying meshes to include some very elongated elements in regions of low velocity gradient, and with negligible effect on the rate of convergence (Palit and Fenner, 1972).

A convergence test was applied after every cycle of overrelaxation. The relative change of each nodal point variable was computed and the magnitude of these changes summed. The convergence criterion used was that this sum should be less than 10^{-5}.

RESULTS

Some typical results are presented for converging rectangular and axisymmetric channels, recirculating flow in rectangular channels, and flow round cylinders. The main object is to demonstrate the power and flexibility of the FE method rather than to make exhaustive studies of particular problems.

Converging Rectangular Channels

Figure 1 shows the converging rectangular channel geometry and for the fixed boundary case considered here $V = 0$. This type of situation is found for example in many polymer extrusion dies. Assuming no slip, $u = v = 0 = \varphi$ on the solid boundaries. The conditions at the inlet and outlet boundaries, $x = 0$ and L respectively, depend on the physical flow situation. For example, the channel might be parallel for $x < 0, x > L$. If this were so then sufficient lengths of these parallel sections should be included to allow fully developed flow to be attained and the appropriate boundary conditions applied. This is a simple matter with the FE method. For present demonstration purposes, however, fully developed flow boundary conditions are applied at $x = 0$ and L.

It is convenient to nondimensionalize the variables in terms of the volumetric flowrate, for example, at $x = 0$,

$$Q = W \int_0^{H_1} u \cdot dy \tag{27}$$

Also a characteristic shear stress τ_c may be defined corresponding to the characteristic shear rate Q/WH_1^2 using the power-law Equation (6):

$$\tau_c = \mu_0 \left(\frac{\gamma_c}{\gamma_0} \right)^{n-1} \gamma_c, \quad \gamma_c = \frac{Q}{WH_1^2} \tag{28}$$

Thus dimensionless variables may be defined as

$$\psi' = W\psi/Q, \quad \varphi' = WH_1\varphi/Q \tag{29}$$

and aspect ratio and compression ratio as

$$A = L/H_1, \quad C = H_1/H_2 \tag{30}$$

The boundary conditions on the lower fixed boundary are $\psi' = \varphi' = 0$, and on the upper boundary $\psi' = -1$, $\varphi' = 0$. The use of fully developed flow boundary conditions at the inlet and outlet implies that the pressures over each are constant, and a dimensionless pressure drop can be defined as

$$P = \frac{p_1 - p_2}{\tau_c} \tag{31}$$

While pressures are not used explicitly in the FE analysis, the required pressure drop may be obtained from Equation (1) by integration along the bottom boundary (Figure 1):

$$p_1 - p_2 = - \int_0^L \left(\frac{\partial \tau_{xy}}{\partial y} \right)_{y=0} \cdot dx \tag{32}$$

The integration is performed numerically, the required stress gradients being obtained in terms of the computed nodal point velocities and viscosities.

Figure 5 shows the results obtained for unit aspect ratio channels for both a Newtonian ($n = 1$) and a non-Newtonian ($n = 0.5$) fluid. Also shown are the corresponding Lubrication Approximation (LA) solutions which assume fully developed flow at every section along the channel. At a typical section where the channel depth is H the LA pressure gradient is given by (see Fenner, 1970)

$$\frac{\partial p}{\partial x} = - \frac{\tau_c}{H} \left(\frac{H_1}{H} \right)^{2n} \cdot 2^{2n+1} \left(\frac{2n}{2n+1} \right)^n \tag{33}$$

hence

$$P = \frac{(C^{2n} - 1)}{n \cdot \tan\alpha} \cdot 2^{2n} \left(\frac{2n}{2n+1} \right)^n \tag{34}$$

where

$$C = (1 - A \tan\alpha)^{-1} \tag{35}$$

Equation (34) is used to plot the LA results in Figure 5.

For $\alpha = 0$ the FE and LA solutions should be identical, which provides a check on FE accuracy. Typically for an 81 (9×9) point mesh this accuracy is of the order of 4%. For such a mesh about 20 cycles of overrelaxation are required for Newtonian flow in a parallel channel, but this number can increase tenfold for large taper angles and highly non-Newtonian materials. A further check on FE accuracy, and one which is particularly useful in tapered channels, is to compute (via the nodal point velocities) the volumetric flow rates at every channel cross-section from the inlet to the outlet. These flow rates should be constant, and any variation is a measure of the accuracy of the solution.

The practical value of expressing the results in dimensionless form is readily seen in Figure 5 in that the single parameter P depends only on the channel geometry (α and A) and the fluid power-law index. Figure 5 also indicates the order of accuracy of the Lubrication Approximation for converging channels. Above a taper angle of say 10° a significant error begins to appear (see Benis, 1967).

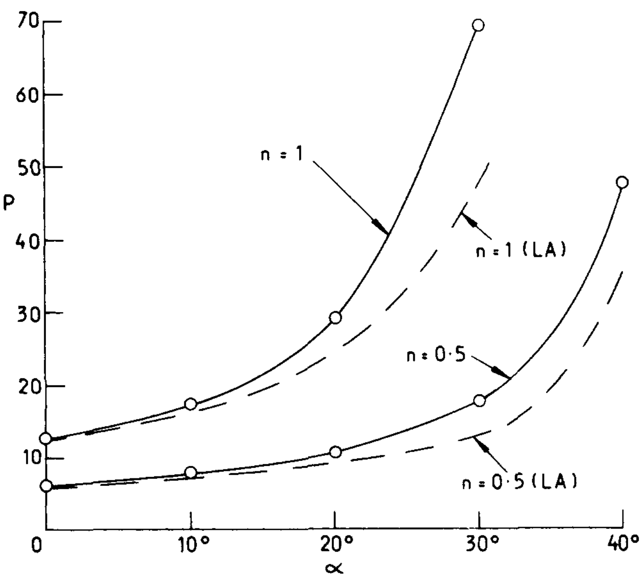

Fig. 5. Dimensionless pressure drops for converging rectangular channels with $A = 1$.

In the present analysis no account is taken of fluid elasticity, whereas converging flows of polymer melts are known to be influenced by the elastic stresses generated. The main purpose is to demonstrate the use of the FE method: given a suitable rheological equation, elasticity could in principle be included. Similar remarks apply to the following results for flow in converging axisymmetric channels.

Converging Axisymmetric Channels

Practical examples of converging channel flows may be axisymmetric and have moving boundaries. Wire coating dies, for instance, can be represented in axial cross-section by Figure 1 where the bottom boundary is the moving wire surface. The flow is symmetric about the axis of the wire. Actual dies usually have several such sections in series with progressively decreasing taper angles (Fenner and Williams, 1967; Fenner, 1970), but such geometries are readily handled with the present method.

This problem is very similar to the rectangular one treated above. Fully developed (axisymmetric) flow boundary conditions are again assumed at the inlet and outlet, but now $\varphi = V$ on the moving boundary. The volumetric flow rate is, for example, at $x = 0$,

$$Q = 2\pi \int_0^{H_1} ur \cdot dy \tag{36}$$

and a characteristic shear stress may be defined corresponding to the characteristic shear rate $Q/2\pi \bar{r} H_1 r_1$, $\bar{r} = \frac{1}{2}(r_0 + r_1)$ being the mean radius at the inlet section:

$$\tau_c = \mu_0 \left(\frac{\gamma_c}{\gamma_0}\right)^{n-1} \gamma_c, \quad \gamma_c = \frac{Q}{2\pi \bar{r} H_1 r_1} \tag{37}$$

Thus dimensionless variables may be defined as

$$\psi' = \frac{2\pi \bar{r}}{Q} \cdot \psi, \quad \varphi' = \frac{2\pi \bar{r} H_1}{Q} \cdot \varphi \tag{38}$$

and aspect ratio as $A = L/r_1$. For axisymmetric geometry a radius ratio parameter $K = r_1/r_0$ is also required. The dimensionless pressure drop is given by Equation (31) and is computed from the axisymmetric equivalent to Equation (32). The boundary conditions on the wire and die surfaces are $\psi' = 0$, $\varphi' = V' = 2\pi \bar{r} H_1 V/Q$ and $\psi' = -1$, $\varphi' = 0$ respectively. In physical terms V' is the ratio between the wire velocity and the mean velocity of the fluid at the inlet.

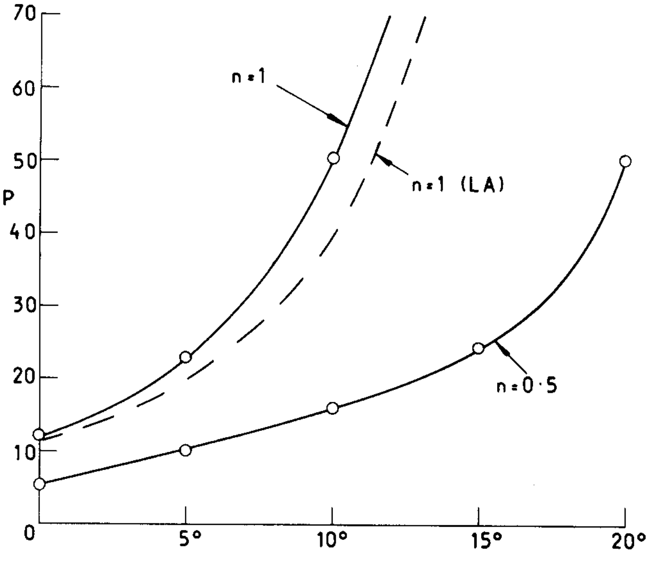

Fig. 6. Dimensionless pressure drops for converging axisymmetric channels with $A = 1$, $K = 2$, $V' = 2$.

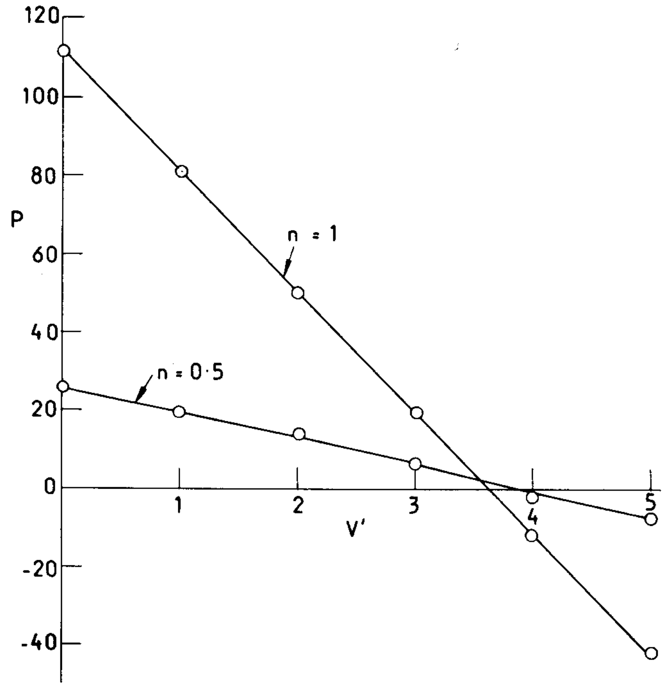

Fig. 7. Dimensionless pressure drops for converging axisymmetric channels with $A = 1$, $K = 2$, $\alpha = 10°$.

Figure 6 shows the results obtained for $A = 1$, $K = 2$, $V' = 2$, $n = 1$ and 0.5, for a range of taper angles, while Figure 7 shows similar results but with $\alpha = 10°$ and a range of V'. Figure 6 also shows the corresponding LA results for $n = 1$ (see Fenner, 1970). At high V' the sign of P is reversed as the effect of the moving boundary is to generate a positive pressure gradient.

Recirculating Channel Flow

Figure 2 shows the geometry and coordinates for recirculating flow in rectangular channels. This is the type of flow which occurs, for example, in the cross-sectional plane of an extruder screw channel (Fenner, 1970). The FE meshes used in this application have nodal points uniformly distributed in rows parallel to the channel boundaries and element arrangements similar to Figure 4.

Whereas in converging channel flows the main interest was in overall pressure drops and flow rates, attention is confined here to streamline patterns. Dimensionless variables may be defined as $\psi' = \psi/HV_x$ and $\varphi' = \varphi/V_x$. Thus, assuming no leakage flow into or out of the channel at its upper corners (Figure 2) ψ' may be defined as zero on all four boundaries, and assuming no slip there $\varphi' = 0$ and 1 on the fixed and moving boundaries respectively.

Figure 8 shows a typical streamline pattern, for $n = 0.5$ and channel shape parameter $S = H/B = 0.5$. The flow is symmetrical about $x = \frac{1}{2}B$, which is in agreement with Burggraf's (1966) findings for slow Newtonian flows. Detailed comparisons with other non-Newtonian solutions are difficult because although this problem has been solved by FD methods (Dyer, 1969; Martin, 1969) it has been included as part of a more general problem. For Newtonian flow, however, Burggraf's FD results for $S = 1$ are more directly comparable. The streamline patterns for this case are identical to within the accuracy of plotting and visual interpolation. The magnitude of the stream

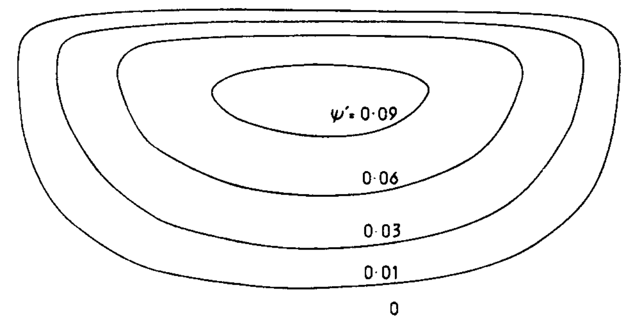

Fig. 8. Streamline pattern for recirculating channel flow with $n = 0.5$, $S = 0.5$.

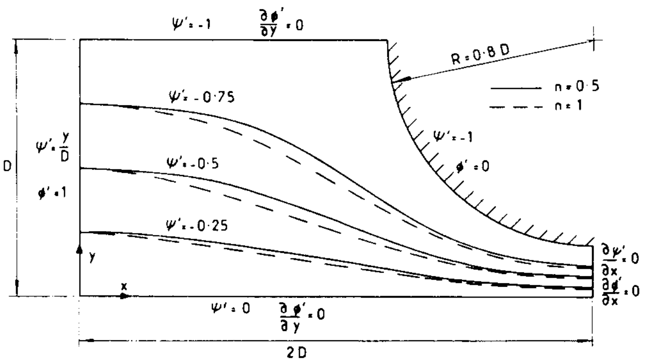

Fig. 9. Streamline pattern for flow round cylinders.

function at the vortex center can, however, be compared numerically. Burggraf gave this value as 0.0998, obtained from both 41×41 and 51×51 point meshes. The present FE method gives values of 0.0951 (9×9 mesh) and 0.0993 (21×21 mesh), indicating the correct convergence. On this basis the accuracy of the FE method is virtually identical to Burggraf's FD method which gave a value of 0.0992 for a 21×21 point mesh.

The recirculating channel flow problem has singularities caused by the velocity discontinuities at the top corners (Figure 2). Such singularities are best treated by using an increased number of elements locally. While this can and has been done, the results presented here are for uniform meshes.

Flow Round Cylinders

As a final example the flow round cylinder is considered. It serves to illustrate the application to more complex geometries which are not readily handled by FD methods, and also the use of derivative boundary conditions. The plane containing the axes of the identical equispaced cylinders is normal to the direction of the undisturbed rectangular flow. A practical example is provided by the 'mixing pins' which are sometimes used in the channels of extruder screws.

Attention can be confined to the typical portion of the flow shown in Figure 9 which includes one quadrant of one cylinder, radius $R = 0.8D$, and has overall dimensions $D \times 2D$, where $2D$ is the pitch of the cylinders. The horizontal length of $2D$ is arbitrary and could be increased to ensure the validity of the assumed undisturbed conditions at the inlet boundary. The other plane boundaries are treated as lines of symmetry. The FE mesh for such a problem is best constructed as a set of nearly equilateral triangles as shown for example by Martin (1968). A much quicker though rather less accurate method is to first generate a rectangular mesh and then to modify the coordinates (in the y-direction) in the region of the cylinder. The dimensionless variables are defined as $\psi' = \psi/UD$ and $\varphi' = \varphi/U$, where U is the free stream velocity (at $x = 0$), and the boundary conditions are shown in Figure 9.

Figure 9 shows the computed streamlines for both Newtonian and a non-Newtonian ($n = 0.5$) fluid. Other flow parameters such as pressure drop and degree of mixing (Fenner, 1970) may also be obtained.

DISCUSSION

The results presented here are typical of a FE analysis of two-dimensional slow non-Newtonian flows. Meshes of triangular spatial elements are used which are very adaptable to problems with complex boundary geometries. A stream function is defined for incompressible two-dimensional flow and is assumed to vary quadratically over each element. This has the advantages of ensuring the correct convergence of the FE solution as the element size is reduced, and resulting in a constant viscosity distribution over each element even for non-Newtonian fluids. This avoids the time consuming necessity of numerically integrating power-law functions over the surfaces of the elements. An additional advantage of using a formulation which results in constant element viscosities is that axisymmetric flow problems can be solved with only trivial modifications to the method for rectangular flows. It is unnecessary to resort to the full cylindrical polar coordinate system. The price which must be paid for these simplifications is a loss of accuracy for a given number of nodal points as compared with more sophisticated formulations. For the problems so far considered, however, the accuracy of the present method using a reasonable number of nodal points is satisfactory for most practical applications. The ultimate test will be through a comparison of computing times for comparable accuracy, but even this must be qualified by a consideration of the relative flexibility and adaptability of the basic computer program to a wide range of practical problems.

In arriving at a formulation which gives rise to constant element viscosities a rather unusual variable is introduced, namely the sum of the two velocity components φ. While it is possible that such a choice may lead to difficulties in certain types of problem, as for example when both components are nonzero on a boundary, no such difficulties have been encountered in the cases so far considered.

Atkinson et al. (1969, 1970) used a FE method to analyze two-dimensional Newtonian flows. While they used triangular elements, they also assumed a modified cubic variation of stream function over each element. As Zienkiewicz (1971) has pointed out, such a formulation may give rise to convergence difficulties as the size of element is reduced. In general, the velocity components are not continuous across the element boundaries, resulting in infinite velocity gradients there, which may invalidate the summation in Equation (19). Atkinson et al. apparently had no such difficulties, and indeed similar incompatible formulations have been successfully employed in other branches of continuum mechanics, (Zienkiewicz, 1971). They studied axisymmetric flow problems, for which it was necessary to introduce the full cylindrical polar coordinate system. Their formulation also resulted in varying velocity gradients over each element which makes it more difficult to apply to non-Newtonian flows than the present method.

The present FE method for two-dimensional flows has some advantages over traditional FD methods. The use of stream function and velocities as variables results in particularly simple boundary conditions. By comparison, FD methods mostly use stream function and vorticity (see, for example, Burggraf, 1966; Gosman et al., 1969; Martin, 1969) for which the boundary conditions are more difficult to apply. The FE method is much more flexible in the type of boundary geometries it can handle as illustrated by the converging channels and flow round cylinders. Whereas Palit and Fenner (1972) showed that long thin triangular elements could be employed with advantage for flows normal to the mesh, in two-dimensional flows the elements should be as equilateral as possible.

The treatment of converging channel flows by traditional FD methods is relatively difficult because of the requirement that the nodal points should lie in rows parallel to the coordinate axes (Gosman et al., 1969). Thus ideally the flow boundaries should also be parallel to these axes. Owing to the linear taper of the converging channels considered here, however, it is possible with a suitable choice of mesh to ensure that nodal points of a rectangular mesh lie on the sloping boundary. Such a method has disadvantages because these points require special treatment and the number of nodal points across the channel depth decreases with depth whereas a roughly constant number is required to maintain accuracy. Alternatively the wedge shapes of rectangular converging channels could be described in cylindrical polar coordinates although the resulting curved inlet and outlet boundaries would be difficult to reconcile with the actual geometry. The treatment of more complex boundary geometries, such as in the flow round cylinders is even more difficult by FD methods.

While attention has been confined here to isothermal two-dimensional flows FE methods can be applied to more general problems. For example, the present method may be combined with the analysis of flow normal to the mesh (Palit and Fenner, 1972) to treat non-Newtonian flow in channels involving both downstream and recirculating flows. Also, heat transfer effects due to thermal conduction and convection and viscous dissipation may be included.

NOTATION

A = aspect ratio
a, b = local nodal point coordinates (Figure 3)
B = channel width (Figure 2)
C = compression ratio
C_1 to C_6 = constants in stream function distribution, Equation (8)
c_{11} to c_{66} = coefficients in Equation (13) for C_1 to C_6
D = half of cylinder pitch (Figure 9)
f = viscous dissipation functional
F, F_x, F_y = stream function and its first partial derivatives with respect to x and y
H = channel depth
H_1, H_2 = channel depths at inlet and outlet (Figure 1)
I_2 = second invariant of rate of deformation tensor
i, j, k = subscripts referencing nodal points
K = radius ratio r_1/r_0
$[K]$ = viscous stiffness matrix
L = length of channel (Figure 1)
m = subscript or superscript referencing element number
N = total number of nodal points
n = power-law index
P = dimensionless pressure drop
p = pressure
p_1, p_2 = pressures at inlet and outlet
Q = volumetric flow rate
R = radius of cylinder
r = radius from axis of symmetry
r_0 = radius of moving boundary (wire)
r_1 = radius at inlet of fixed boundary (die)
$\overline{r}$ = a mean radius
r_m = radius of centroid of element m
S = channel shape parameter H/B
U = free stream velocity
u, v = velocity components in the x and y directions
V = channel boundary velocity (Figure 1)
V_x = channel boundary velocity (Figure 2)
W = width of flow normal to x-y plane
x, y = Cartesian coordinates

Greek Letters

α = channel taper angle (Figure 1)
γ_0 = reference shear rate
γ_c = characteristic shear rate
Δ = element area
∇^2, ∇^4 = harmonic and biharmonic operators
μ = viscosity
μ_0 = effective viscosity at γ_0
τ = viscous stress
τ_c = characteristic viscous shear stress
φ = sum of velocity components $u + v$
φ' = dimensionless φ
χ = integral of functional f
ψ = stream function
ψ' = dimensionless stream function
ω = vorticity

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